Compact Kähler threefolds with nef anticanonical line bundle
Lecture 1

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compact Kähler manifold with nef anticanonical line bundle

Conjecture

Let X be a compact Kähler manifold with the nef anti-canonical bundle $-K_X$. Then, there exists a fibration $\varphi : X \to Y$ with the following:

- $\varphi : X \to Y$ is a locally trivial fibration;
- Y is a compact Kähler manifold with $c_1(Y) = 0$;
- F, which is the fiber of $\varphi : X \to Y$, is rationally connected (i.e. any two general points of F are in the image of some holomorphic map $\mathbb{P}^1 \to F$).

Known in projective case by Cao-Höring.
But widely open in the compact Kähler case.

Main result

Theorem, Matsumura-Wu '23

Let X be a non-projective compact Kähler 3-fold with nef anti-canonical bundle. Then X admits a finite étale cover that is one of the following:

- a compact Kähler manifold with vanishing first Chern class;
- the product of a K3 surface and the projective line $\mathbb{P}^1$;
- the projective space bundle $\mathbb{P}(E)$ of a numerical flat vector bundle E of rank 2 over a 2-dimensional (compact complex) torus.

This implies the structure conjecture in the case of $\dim X = 3$.

3-dim compact Kähler MMP

Theorem, Höring-Peternell 16

Let X be a $\mathbb{Q}$-factorial compact Kähler space of dimension 3 with terminal singularities. Assume that $\dim R(X) = 2$, where $R(X)$ is the base of an MRC fibration $X \dashrightarrow R(X)$ of X. Then, we have that X is bimeromorphic to a MF (Mori fiber) space; more precisely, there exist a bimeromorphic map $\pi : X \dashrightarrow X'$ and a MF space $\varphi : X' \to S$ such that

(a) $X \dashrightarrow X'$ is obtained from the composition of divisorial contractions and flips;

(b) X' is a $\mathbb{Q}$-factorial compact Kähler space with terminal singularities;

(c) S is a $\mathbb{Q}$-factorial compact Kähler space of dimension 2 with klt singularities;

3-dim compact Kähler MMP

(d) S is non-uniruled and K_S is pseudo-effective;

(e) $-K_{X'}$ is φ-ample and the relative Picard number $\rho(X'/S)$ is 1;

(f) $\varphi : X' \to S$ is equi-dimensional and of relative dimension 1.

Key point

$-K_{X'}$ is φ-ample. In 3-dim compact Kähler non-projective case, the MMP consists of only one step of Mori fiber (for short, MF) space such that $-K_X$ is relative ample.

Bad news: X' may be singular a priori...

Example

Example (A global $\mathbb{Q}-$conic bundle)

For a Kummer surface $S := A/\mu_2$ with a torus A of dimension 2, we consider

$$X' := (\mathbb{P}^1 \times A)/\mu_2 \to S = A/\mu_2,$$

where μ_2 acts on $\mathbb{P}^1 \times A$ by $-1 \cdot (t, z_1, z_2) = (-t, -z_1, -z_2)$. Both S and X' are simply connected and $\varphi : X' \to S$ is a $\mathbb{Q}$-conic bundle such that $-K_{X'}$ is nef. However X' is not outcome of MMP for some smooth X with $-K_X$ nef (cf. Peternell-Serrano).

Difficulty: For A general, X' has no hypersurface. Attention is needed to contradict by intersection numbers.

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Notations

We abbreviate singular Hermitian metrics to “metrics.” The term of “fibrations” denotes a proper surjective morphism with connected fibers, the term of “analytic varieties” denotes an irreducible and reduced complex analytic space, the term of “Kähler spaces” denotes a normal analytic variety admitting a Kähler form (i.e., a smooth $(1,1)$-form on X which is the restriction of $i\partial\overline{\partial}$ of some smooth strictly psh function of the ambiant space).

Ph functions

Let X be a normal analytic variety. A pluriharmonic function on X can be locally written as the real part of a holomorphic function, that is, the kernel of the $\partial\overline{\partial}$-operator on the sheaf of distributions of bidegree $(0,0)$ coincides with the sheaf $\mathbb{R}\mathcal{O}_X$ of real parts of holomorphic functions (e.g., see Lemma 4.6.1, Boucksom-Eyssidieux-Guedj, BEG for short).

Bott-Chern cohomology group

Then, the Bott-Chern cohomology group of X is defined by

$$H^{1,1}_{BC}(X, \mathbb{C}) := H^1(X, \mathbb{R}\mathcal{O}_X).$$

A smooth $(p, q)-$form on X is the local restriction of some smooth $(p, q)-$form of the ambiant space.
The short exact sequence

$$0 \to \mathbb{R}\mathcal{O}_X \to C^\infty_X \to C^\infty_X/\mathbb{R}\mathcal{O}_X \to 0$$

induces

$$H^0(X, C^\infty_X/\mathbb{R}\mathcal{O}_X) \to H^1(X, \mathbb{R}\mathcal{O}_X).$$

Thus a Kähler form defines a Kähler class in $H^1(X, \mathbb{R}\mathcal{O}_X)$.

Bott-Chern cohomology

A $(1,1)$-form with local potentials on X is defined to be a section of the quotient sheaf $C_X^\infty/\mathbb{R}\mathcal{O}_X$. A $(1,1)$-form with local potentials can be more concretely described as a closed $(1,1)$-form on X that is locally of the form $i\partial\overline{\partial}u$ for a smooth function u.
The first Chern class $c_1(L) \in H^{1,1}_{BC}(X, \mathbb{C})$ of a line bundle L on X is defined by the Bott-Chern cohomology class of $(\sqrt{-1}/2\pi)\Theta_h(L)$, where $\Theta_h(L)$ denotes the Chern curvature of a smooth metric h on L (e.g., which is constructed by a partition of unity). Note that $c_1(L)$ does not depend on the choice of smooth metrics and the first Chern class of $\mathbb{Q}$-Cartier divisors can be also defined by linearity.

[Proposition 4.6.3]BEG

Let $\alpha \in H^{1,1}_{BC}(X, \mathbb{C})$ be a Bott-Chern cohomology class on a normal analytic variety X, and let T be a positive current on X_{reg} representing the restriction $\alpha|_{X_{\mathrm{reg}}} \in H^{1,1}_{BC}(X_{\mathrm{reg}}, \mathbb{C})$. Then, the current T is uniquely extended to the positive current with local potential on X representing $\alpha \in H^{1,1}_{BC}(X, \mathbb{C})$.
In particular, we can often work on the regular part.

Technical Remark

transformation of Kähler class

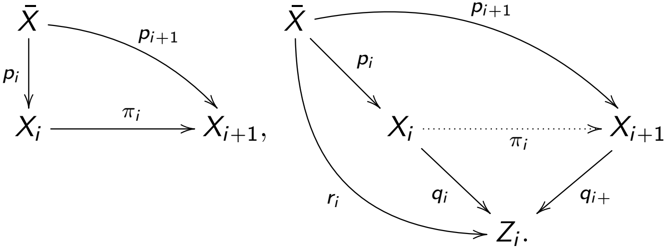
Let $\pi : X \to X'$ be a modification of analytic variety. Let ω be a Kähler class on X. $\pi_*\omega$ does not define necessarily a Bott-Chern class (due to non-existence of local potential.)

Assume that π is a blow up with exceptional divisor E and ω' be a Kähler form on X'. We should consider $-K_X + \epsilon(\pi^*\omega' - \delta E)$ for $0 < \delta \ll 1$ such that $\pi_*(-K_X + \epsilon(\pi^*\omega' - \delta E)) = -K_{X'} + \epsilon\omega'$ defines a Bott-Chern class.

Assume that X compact Kähler threefold. The bimeromorphic map $X \dashrightarrow X'$ in MMP is decomposed as follows:

$$X =: X_0 \dashrightarrow X_1 \dashrightarrow \cdots \dashrightarrow X_N := X', \qquad (1)$$

where each bimeromorphic map $\pi_i : X_i \dashrightarrow X_{i+1}$ is a divisorial contraction or flip. Let $\bar{X}$ be a compact Kähler manifold with a bimeromorphic morphism $p_i : \bar{X} \to X_i$ that resolves the indeterminacy locus of π_i (when π_i is a flip).



Note that Z_i and X_i are Kähler spaces.

Then, for any $0 \leqslant i \leqslant N$, there exists a Kähler form ω_i on X_i such that the Bott-Chern class

$$\{p_{0*}(p_{i+1}^*\omega_{i+1} - p_i^*\omega_i)\} + O(E, K_X)$$

is represented by a positive current that is smooth on the biholomorphic locus of $X \dashrightarrow X'$, where $O(E, K_X)$ is a linear combination of the first Chern classes of K_X and the exceptional divisors. In particular, the Bott-Chern cohomology class $\{p_{0*}p_i^*\omega_i - \omega_0\} + O(E, K_X)$ is represented by a positive current that is smooth on the biholomorphic locus of $X \dashrightarrow X'$.

The varieties X_i are biholomophic among them on a non-empty Zariski open set, we will call this open set biholomorphic locus which will be seen as open subset of all X_i's. Note that the complement of biholomorphic locus in X' is of codimension at least 2.

Let ω_i be a Kähler form on X_i. There exists a positive current $T_\varepsilon \in c_1(-K_{X_i}) + \varepsilon\{\omega_i\}$ such that T_ε is smooth on the biholomorphic locus of $X \dashrightarrow X'$.

Psh functions

Définition 1.9, Demailly '85:
Let $V : Z \to [-\infty, \infty[$ be a function that is not identically infinite over any open set of Z. Then V is called psh (resp. quasi-psh) if for any local embedding $j : Z \hookrightarrow \Omega \subset \mathbb{C}^N$, V is the local restriction of a psh (resp. quasi-psh) function on Ω.
We have the following equivalent definition of psh functions due to J.E. Fornaess and R. Narasimhan.
A function V is a psh function over an analytic variety X if and only if
(1) V is upper semi-continuous.
(2) for any holomorphic map $f : \Delta \to X$ from the unit disc Δ, either $V \circ f$ is subharmonic or $V \circ f$ is identically infinite.

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Singular Hermitian metrics on torsion-free sheaves on normal analytic varieties

Let $\mathcal{E}$ be a torsion-free coherent sheaf on a normal analytic variety X. *A singular Hermitian metric h* on $\mathcal{E}$ is a possibly singular Hermitian metric on the vector bundle $\mathcal{E}|_{X_0}$. Here $\mathcal{E}|_{X_0}$ is the restriction of $\mathcal{E}$ to $X_0 := X_{\mathrm{reg}} \cap X_{\mathcal{E}}$, where X_{reg} is the non-singular locus of X and $X_{\mathcal{E}}$ is the maximally locally free locus of $\mathcal{E}$. Note that $X_0 \subset X$ is a Zariski open set with $\operatorname{codim}(X \backslash X_0) \geqslant 2$. A singular metric on a vector bundle is locally a measurable map to the space of Hermitian matrix satisfying $0 < \det h < \infty$ almost everywhere (compatible with the transition functions).

Weak positivity

For a smooth $(1,1)$-form θ on X with local potential, we write as

$$\sqrt{-1}\Theta_h \geqslant \theta \otimes \mathrm{id} \text{ on } X$$

if the function $\log |e|_{h^*} - f$ is psh for any local section e of $\mathcal{E}^*$, where f is a local potential of θ (i.e., $\theta = \sqrt{-1}\partial\bar{\partial} f$) and h^* is the induced metric on the dual sheaf $\mathcal{E}^* := Hom(\mathcal{E}, \mathcal{O}_X)$. The plurisubharmonicity can be extended through a Zariski closed set of codimension $\geqslant 2$; therefore it is sufficient to check that $\log |e|_{h^*} - f$ is a psh function on an open set of X_0 by $\mathrm{codim}(X \backslash X_0) \geqslant 2$.

Weak positivity

Let X be a Kähler space, ω_X be a Kähler form on X, and θ be a $(1,1)$-form on X with local potential. A torsion-free sheaf $\mathcal{E}$ on X is said to be *θ-weakly positively curved* if there exist singular Hermitian metrics $\{h_\varepsilon\}_{\varepsilon>0}$ on $\mathcal{E}$ such that $\sqrt{-1}\Theta_{h_\varepsilon} \geqslant (\theta - \varepsilon\omega_X) \otimes \mathrm{id}$ on X. We simply say that $\mathcal{E}$ is weakly positively curved in the case of $\theta = 0$.

Positivity of direct image

Let $\varphi : X \to Y$ be a fibration between (not necessarily compact) Kähler manifolds (X, ω_X) and (Y, ω_Y). Let L be a line bundle on X and θ be a d-closed $(1,1)$-form on Y.

(a) The non-nef locus of $-K_{X/Y}$ is not dominant over Y in the following sense: $-K_{X/Y}$ has singular metrics $\{g_\delta\}_{\delta>0}$ such that $\sqrt{-1}\Theta_{g_\delta} \geqslant -\delta\omega_X$ holds on X and the upper-level set $\{x \in X \mid \nu(g_\delta, x) > 0\}$ of Lelong numbers is not dominant over Y, where $\nu(g_\delta, x)$ is the Lelong number of a local potential of g_δ at x;

(b) L is a φ-big line bundle in the following sense: L has a singular Hermitian metric g such that $\sqrt{-1}\Theta_g + \varphi^*\omega_Y \geqslant \omega_X$ holds on X;

(c) L is $\varphi^*\theta$-weakly positive in the following sense: L has singular metrics $\{h_{\delta'}\}_{\delta'>0}$ such that $\sqrt{-1}\Theta_{h_{\delta'}} \geqslant \varphi^*\theta - \delta'\omega_X$ on X.

Positivity of direct image

Then, we have:

(1) The direct image sheaf $\varphi_*(-mK_{X/Y} + L)$ is $((1-\varepsilon)\theta - \varepsilon\omega_Y)$-positively curved for any $m \in \mathbb{Z}_+$ and $\varepsilon > 0$.

(2) If we further assume that $\omega_Y \geqslant \theta$ holds, then $\varphi_*(-mK_{X/Y} + L)$ is θ-weakly positively curved. In particular, if L is a pseudo-effective line bundle, then $\varphi_*(-mK_{X/Y} + L)$ is weakly positively curved.

proof

The sheaf can be regarded as the direct image sheaf of the pluri-adjoint bundle

$$-mK_{X/Y} + L = kK_{X/Y} \overbrace{-(m+k)K_{X/Y}}^{\text{with } g_\delta^{m+k}} + \overbrace{L}^{\text{with } g^\varepsilon \cdot h_{\delta'}^{1-\varepsilon}} .$$

Let us consider the curvature current and multiplier ideal sheaf associated to the metric

$$G := g_\delta^{m+k} \cdot g^\varepsilon \cdot h_{\delta'}^{1-\varepsilon} \text{ on } -(m+k)K_{X/Y} + L.$$

We can easily confirm that

$$\mathcal{I}(G^{1/k})|_{X_y} = \mathcal{I}\big(g_\delta^{m/k+1} \cdot g^{\varepsilon/k} \cdot h_{\delta'}^{(1-\varepsilon)/k}\big)|_{X_y} = \mathcal{O}_{X_y} \qquad (2)$$

holds for a very general fiber X_y and a sufficiently large $k \gg 1$ (which depends on δ', but not depend on ε and δ).

Proof

Indeed, by Condition (a), we have that $\nu(g_\delta, x) = 0$ for any $x \in X_y$ since X_y is a very general fiber. Hence, we obtain

$$\nu(g_\delta^{m/k+1} \cdot g^{\varepsilon/k} \cdot h^{(1-\varepsilon)/k}, x) \leqslant \nu(g^{1/k} \cdot h_{\delta'}^{1/k}, x) < 1$$

for $k \gg 1$. Then, Skoda's lemma shows that $\mathcal{I}(G^{1/k})|_{X_y} = \mathcal{O}_{X_y}$; hence the natural inclusion

$$\varphi_*((-mK_{X/Y} + L) \otimes \mathcal{I}(G^{1/k})) \to \varphi_*(-mK_{X/Y} + L) \qquad (3)$$

is generically surjective. We have

$$\sqrt{-1}\Theta_G \geqslant \big(\varepsilon - \delta(m+k) - \delta'\big)\omega_X - \varepsilon\varphi^*\omega_Y + (1-\varepsilon)\varphi^*\theta.$$

For a given $\varepsilon > 0$, after taking $\delta' > 0$ with $\delta' < (1/2)\varepsilon$, we fix a sufficiently large k satisfying (2). Furthermore, we take $1 \gg \delta > 0$ so that $\delta(m+k) < (1/2)\varepsilon$.

Cao-Höring

Let $\varphi : X \to Y$ be an equi-dimensional fibration between compact Kähler spaces X and Y with Kähler forms ω_X and ω_Y. Let $Y_0 \subset Y$ be a Zariski open set with $\operatorname{codim}(Y \backslash Y_0) \geqslant 2$ such that $X_0 := \varphi^{-1}(Y_0)$ and Y_0 are smooth and that $\varphi_0 := \varphi|_{X_0} : X_0 \to Y_0$ is a smooth fibration. Let L be a line bundle on X. Assume the following conditions:

(a) $-K_X$ is $\mathbb{Q}$-Cartier and the non-nef locus of $-K_X$ is not dominant over Y in the sense of Condition (a);

(b) $-K_Y$ is $\mathbb{Q}$-Cartier and numerically trivial;

(c) L is a pseudo-effective and φ-ample line bundle on X;

Cao-Höring

Let r be the rank of $\varphi_*(L)$ and p be a sufficiently large integer with $p/r \in \mathbb{Z}_+$. Define the sheaf $\mathcal{V}_p$ on Y by

$$\mathcal{V}_p := \varphi_*(pL) \otimes \left(\frac{p}{r} \det \varphi_*(L)\right)^*.$$

Then, both $\mathcal{V}_p$ and $(\det \mathcal{V}_p)^*$ are weakly positively curved.
Key point: "Positive"-"Positive" is still "Positive".

Proof

Calim 1: By positivity of direct image, $\varphi_*(pL)$ is weakly positively curved on Y for any $p \in \mathbb{Z}_+$.
Claim 2: Let r_p be the rank of $\varphi_*(pL)$. Then, the sheaf

$$r_p pL \otimes \left(\varphi^* \det \varphi_*(pL)\right)^*$$

is weakly positively curved on X.
Let Z be the $r = r_p$-times fiber product $X \times_Y X \times_Y \cdots \times_Y X$ with the i-th projection $\mathrm{pr}_i : Z \to X$ and the natural morphism $\psi : Z \to Y$:

$$\begin{array}{ccc} Z & \xrightarrow{\ \mathrm{pr}_j\ } & X \\ {\scriptstyle \mathrm{pr}_i}\downarrow & \searrow^{\psi} & \downarrow{\scriptstyle \varphi} \\ X & \xrightarrow[\ \varphi\ ]{} & Y. \end{array}$$

proof

Assume $p = 1$. Set

$$L_r := \sum_{i=1}^{r} \mathrm{pr}_i^* L \text{ and } L' := L_r \otimes (\psi^* \det \varphi_*(L))^*.$$

By Condition (c), we obtain a smooth metric g on L such that $\sqrt{-1}\Theta_g + \varphi^*\omega_Y \geqslant \omega_X$ holds on X. Let us consider the smooth metric

$$G := (\sum_{i=1}^{r} \mathrm{pr}_i^* g) \cdot (\psi^* g_1)^{-1} \text{ on } L' = (\sum_{i=1}^{r} \mathrm{pr}_i^* L) \otimes (\psi^* \det \varphi_*(L))^*,$$

where g_1 is a smooth metric on $\det \varphi_* L|_{Y_0}$. Then, we obtain that

$$\sqrt{-1}\Theta_G(L') + \sum_{i=1}^{r} \mathrm{pr}_i^* \varphi^* \left(\omega_Y + \frac{1}{r}\sqrt{-1}\Theta_{g_1}\right) \geqslant \sum_{i=1}^{r} \mathrm{pr}_i^* \omega_X \text{ on } Z_0.$$

proof

There exists the (non-zero) natural morpshim

$$\det \varphi_*(L) \to (\varphi_*(L))^{\otimes r} \cong \psi_*(L_r) \text{ on } Y_0,$$

which shows that $h^0(Z_0, L') \neq 0$ by the definition of L'. In particular, the line bundle $L'|_{Z_0}$ satisfies Condition (c). Positivity of direct image applies to $\psi_* L'$.

$$\psi^* \psi_*(L') \to L' \text{ on } Z_0,$$

which is generically surjective by $h^0(Z_0, L') \neq 0$. Let G_ε be the metric on $L'|_{Z_0}$ induced by positivity of direct image and the above morphism. We identify the diagonal subset Δ of the fiber product Z_0 with X_0. This finishes the proof of Claim 2.

proof

Claim 2 +positivity of direct image imply

$$\varphi_* L \otimes \left(\frac{1}{pr_p} \det \varphi_*(pL) \right)^* \text{ is weakly positively curved} \qquad (4)$$

Take determinant and for p large enough. $(\det \mathcal{V}_p)^*$ is pseudo-effective by Claim 1.
Since L is φ-ample, the natural morphism

$$\mathcal{W}_p := \mathrm{Sym}^p(\varphi_* L) \otimes \left(\frac{p}{r} \det \varphi_* L \right)^* \to \varphi_*(pL) \otimes \left(\frac{p}{r} \det \varphi_* L \right)^* = \mathcal{V}_p$$

is generically surjective for $p \gg 1$. $\mathcal{V}_p$ is also weakly positively curved.

Thank you for your attention!