Compact Kähler threefolds with nef anticanonical line bundle
Lecture 2

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Singular Hermitian metrics on torsion-free sheaves on normal analytic varieties

Let $\mathcal{E}$ be a torsion-free coherent sheaf on a normal analytic variety X. *A singular Hermitian metric h* on $\mathcal{E}$ is a possibly singular Hermitian metric on the vector bundle $\mathcal{E}|_{X_0}$. Here $\mathcal{E}|_{X_0}$ is the restriction of $\mathcal{E}$ to $X_0 := X_{\mathrm{reg}} \cap X_{\mathcal{E}}$, where X_{reg} is the non-singular locus of X and $X_{\mathcal{E}}$ is the maximally locally free locus of $\mathcal{E}$. Note that $X_0 \subset X$ is a Zariski open set with $\mathrm{codim}(X \backslash X_0) \geqslant 2$. A singular metric on a vector bundle is locally a measurable map to the space of Hermitian matrix satisfying $0 < \det h < \infty$ almost everywhere (conpatible with the transition functions).

Weak positivity

For a smooth $(1,1)$-form θ on X with local potential, we write as

$$\sqrt{-1}\Theta_h \geqslant \theta \otimes \mathrm{id} \text{ on } X$$

if the function $\log |e|_{h*} - f$ is psh for any local section e of $\mathcal{E}^*$, where f is a local potential of θ (i.e., $\theta = \sqrt{-1}\partial\bar{\partial} f$) and h^* is the induced metric on the dual sheaf $\mathcal{E}^* := Hom(\mathcal{E}, \mathcal{O}_X)$. The plurisubharmonicity can be extended through a Zariski closed set of codimension $\geqslant 2$; therefore it is sufficient to check that $\log |e|_{h*} - f$ is a psh function on an open set of X_0 by $\mathrm{codim}(X \backslash X_0) \geqslant 2$.

Weak positivity

Let X be a Kähler space, ω_X be a Kähler form on X, and θ be a $(1,1)$-form on X with local potential. A torsion-free sheaf $\mathcal{E}$ on X is said to be *θ-weakly positively curved* if there exist singular Hermitian metrics $\{h_\varepsilon\}_{\varepsilon>0}$ on $\mathcal{E}$ such that $\sqrt{-1}\Theta_{h_\varepsilon} \geqslant (\theta - \varepsilon\omega_X) \otimes \mathrm{id}$ on X. We simply say that $\mathcal{E}$ is weakly positively curved in the case of $\theta = 0$.

Strongly pseudo-effective vector bundle

Advantage of weakly positively curved sheaf:
Defined for non-necessarily locally free sheaf over non necessarily smooth space.
Disadvantage of weakly positively curved sheaf:
Difficult to study second Chern class (No definition in general!)
In general, the direct image is reflexive under flat morphism. To show the locally-freeness, one usually use Bando-Siu's result stating that a stable reflexive sheaf with vanishing first Chern class over a compact Kähler manifold is a projectively flat vector bundle if the second Chern class is trivial.

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Nef vector bundle

Definition. (Hartshorne, '66)

Let X be a projective manifold and E a holomorphic vector bundle on X. E is called ample if and only if $\mathcal{O}_{\mathbb{P}(E)}(1)$ is an ample line bundle.

Definition. (DPS, '94)

Let (X, ω) be a compact Kähler manifold and E a holomorphic vector bundle on X. E is called nef if and only if $\mathcal{O}_{\mathbb{P}(E)}(1)$ is a nef line bundle (i.e.$\forall \varepsilon > 0$, there exists a smooth metric $(\mathcal{O}_{\mathbb{P}(E)}(1), h_\varepsilon)$ such that the Chern curvature representing $c_1(\mathcal{O}_{\mathbb{P}(E)}(1))$ satisfies $i\Theta(\mathcal{O}_{\mathbb{P}(E)}(1), h_\varepsilon) \geqslant -\varepsilon \pi^* \omega$) where $\pi : \mathbb{P}(E) \to X$.

Strongly psef vector bundle

Pseudo-effective line bundle

A line bundle L over a compact manifold X is called psef if $\exists T \geqslant 0 \in c_1(L)$ in the sense of currents.

Definition. BDPP, '13

Let (X, ω) be a compact Kähler manifold and E a holomorphic vector bundle on X. Then E is said to be strongly pseudo-effective (strongly psef for short) if the line bundle $\mathcal{O}_{\mathbb{P}(E)}(1)$ is pseudo-effective on the projectivized bundle $\mathbb{P}(E)$ of hyperplanes of E, i.e. if for every $\varepsilon > 0$ there exists a singular metric h_ε with analytic singularities on $\mathcal{O}_{\mathbb{P}(E)}(1)$ and a curvature current $i\Theta(h_\varepsilon) \geqslant -\varepsilon\pi^*\omega$, and if the projection $\pi(\mathrm{Sing}(h_\varepsilon))$ of the singular set of h_ε is not equal to X.

Pseudo-effective vector bundle

Equivalent definition, BDPP, '13

Let X be a projective manifold. A holomorphic vector bundle E on X is pseudo-effective if and only if for any given ample line bundle A on X and any positive integers m_0, p_0, the vector bundle

$$S^p((S^m E) \otimes A)$$

is generically generated (i.e. generated by its global sections on the complement $X \backslash Z_{m,p}$ of some proper algebraic set $Z_{m,p} \subset X$) for some [resp. every] $m \geqslant m_0$ and $p \geqslant p_0$.

Regularisation

Any $T \geqslant 0$ in the sense of currents is locally limit of smooth positive forms.

(Global) Regularisation, Demailly, '82

let $T = \theta + i\partial\overline{\partial}\varphi$ be a closed $(1,1)$-current, where θ is a smooth form. Suppose that a smooth $(1,1)$-form γ is given such that $T \geqslant \gamma$. Then there exists a decreasing sequence of smooth functions φ_k converging to φ such that, if we set $T_k := \theta + i\partial\overline{\partial}\varphi_k$, we have

(1) $T_k \to T$ weakly,

(2) $T_k \geqslant \gamma - C\lambda_k\omega$, where $C > 0$ is a constant depending on (X, ω) only, and λ_k is a decreasing sequence of continuous functions such that $\lambda_k(x) \to \nu(T, x)$ for all $x \in X$.

Regularisation

(Global) Regularisation, Demailly, '92

let $T = \theta + i\partial\overline{\partial}\varphi$ be a closed $(1,1)$-current, where θ is a smooth form. Suppose that a smooth $(1,1)$-form γ is given such that $T \geqslant \gamma$. Then there exists a decreasing sequence of quasi-psh functions φ_k converging to φ such that, if we set $T_k := \theta + i\partial\overline{\partial}\varphi_k$, we have

(1) $T_k \to T$ weakly,

(2) φ_k is locally given by $c \log \sum_i |g_i|^2 + O(1)$ where $c \geqslant 0$, g_i are local holomorphic functions and $O(1)$ is bounded. (We say that φ_k has analytic singularities.)

(3) $T_k \geqslant \gamma - \varepsilon_k \omega$ in the sense of currents, where ε_k is a decreasing sequence such that $\varepsilon_k \to 0$.

Formal property

Proposition,-22

E strongly psef $\Rightarrow \det(E)$ is psef.
E strongly psef $\iff$ for some $m > 0$, $S^m E$ strongly psef.
For surjective bundle morphism $E \to Q$, E strongly psef $\Rightarrow Q$ strongly psef
E, F strongly psef $\Rightarrow E \oplus F$, $E \otimes F$ strongly psef.

Example

$E = \oplus L_i$ nef/strongly psef $\iff \forall i$, L_i nef/psef.
$\mathcal{O}_{\mathbb{P}(E)}(1)$ is big/psef $\iff \exists i_0$ s.t. L_{i_0} is big/psef.

Comparison

Finsler metric is a continuous nonnegative function $F : E \to [0, \infty[$ defined on the vector bundle so that for each point x, $v \in E_x$

$F(\lambda v) = |\lambda| F(v)$ for all $\lambda \in \mathbb{C}$ (homogeneity).

$F(v) > 0$ unless $v = 0$ (positive definiteness).

Hermitian metric is Finsler.
Metric on $\mathcal{O}_{\mathbb{P}(E)}(1)$ is equivalent to Finsler metric on E^*.
$\mathcal{O}_{\mathbb{P}(E)}(1)$ is an ample line bundle if and only if E^* carries a smooth Finsler metric which is strictly plurisubharmonic on the total space $E^* \backslash \{0\}$.

Comparison

Griffiths Conjecture

Ampleness of E is equivalent to the existence of a Griffiths positive hermitian metric, thus to the existence of a hermitian strictly plurisubharmonic metric on E^*.

In other words, how to construct Griffiths positive Hermitian metric from the Finsler metric?
Similarly, a weakly positively curved vector bundle is strongly pseudoeffective.
Conversely, it is conjectured to be true as Griffiths type conjecture.
Known in rank 1 case or over base of dimension 1 (Wu'22).

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Background

If E nef, the Segre classes

$$s_i(E) := \pi_*(c_1(\mathcal{O}_{\mathbb{P}(E)}(1))^{r-1+i})$$

contain a closed positive current where E is of rank r and $\pi : \mathbb{P}(E) \to X$ is the projection. (See e.g. DPS '94)
Note that $c_1(E) = s_1(E), s_2(E) = c_1(E)^2 - c_2(E)$.
What happens if E is strongly pseudoeffective?

Segre current

Example

Consider X the blow up $\mathbb{P}^2$ at a point with exceptional divisor E. The closed positive current associated to E denoted by $[E]$ does not well define $[E] \wedge [E]$ as closed positive current representing the correct cohomology class since $\{[E]\}^2 = -1$.

Theorem,Demailly, agbook Chap. III.4

Let $T_1, \cdots, T_r$ be closed positive currents with analytic singularities such that $\forall i_1 < \cdots < i_m$, the codimension of $\cap_{i_j} Sing(T_{i_j})$ is at least m. Then $T_1 \wedge \cdots \wedge T_r$ is well-defined as positive current and represents the cohomology class $\{T_1\} \wedge \cdots \wedge \{T_r\}$.

Segre current

Question

In the relative situation (i.e. a proper submersion $pi : X \to Y$ between compact Kähler manifolds of relative dimension $r - 1$), how to define $\pi_*(T_1 \wedge \cdots \wedge T_r)$ given weak codimension condition on $\pi(Sing(T_i))$?

This question appears previously in LRRS'18 without estimate of Lelong number.

Segre current

Theorem,-22

In the relative situation, assume:
(1) (codimension condition) T is a closed positive $(1,1)-$current in the cohomology class $\{\alpha\} \in H^{1,1}(X,\mathbb{R})$ such that T has analytic singularities and is smooth on $X \backslash \pi^{-1}(Z)$ with Z a closed analytic set of codimension at least k.
(2) (existence of local reference potential) for any $y \in Y$, there exist an open neighborhood U of y and a quasi-psh function ψ on X such that $\alpha + i\partial\overline{\partial}\psi \geqslant 0$ in the sense of currents on $\pi^{-1}(U)$ and ψ is smooth outside a closed analytic set of codimension at least $k+r$.
Then there exists a closed positive current in the cohomology class $\pi_* \alpha^{r+k-1}$ with multiplicity estimate.

Segre current

Construction

Let ψ be a local reference potential. Then the Monge-Ampère operator $(\alpha + i\partial\overline{\partial}\log(e^{\varphi} + \delta e^{\psi}))^{r-1+k}$ is well defined for every $\delta > 0$ with the codimension condition. By weak compactness,

$$\pi_*(\alpha + i\partial\overline{\partial}\log(e^{\varphi} + \delta_\nu e^{\psi}))^{r-1+k}$$

which all belong to the cohomology class $\pi_*\alpha^{r-1+k}$, has a weak limit as $\delta_\nu \to 0$ for some subsequence.

Difficulty

(1) Such ψ is not global positive (i.e. $\alpha + i\partial\overline{\partial}\psi$ is not necessarily positive).
(2) The limit is not necessarily unique a priori.

Segre current

Note that (2) implies that the limit is positive since it is positive on the open set where ψ is defined.

Proposition,ABW19 BI19, -22

Let φ be a quasi-psh function with analytic singularities over on a (connected) complex n-dimensional manifold X, and $u \in C^\infty(X)$. Then for any exponent p ($1 \leqslant p \leqslant n$), the asymptotic limit of Monge-Ampère operator $\lim_{\delta \to 0}(i\partial\overline{\partial}\log(e^\varphi + \delta e^u))^p$ is always well defined as a current (but not necessarily positive, even when $i\partial\overline{\partial}\varphi \geqslant 0$, and the limit may depend on u).

Segre current

Denote by T_1, T_2 the limit currents obtained with ψ_1 and ψ_2. Assume that A' is the union of the singular loci of ψ_1 and ψ_2. By assumption, $\pi(A')$ is of codimension at least $k+1$ in X. Then $T_1 - T_2$ is a normal (k,k)-current supported in $\pi(A) \cup \pi(A')$ by the continuity of Bedford-Taylor operator.
The support theorem yields

$$T_1 - T_2 = \sum_{\nu} c_\nu [Z_\nu]$$

where Z_ν are the codimension k irreducible components of $\pi(A)$ and $c_\nu \in \mathbb{R}$.
Take a local cut-off function θ and prove (to show that $c_\nu = 0$) that

$$\lim_{\delta \to 0} \int_X \left(\pi_* T_{1,\delta}^{k+r-1} - \pi_* T_{2,\delta}^{k+r-1} \right) \wedge \theta \omega^{n-k} = 0.$$

Segre current

A direct calculation shows that
$\int_X \left(\pi_* T_{1,\delta}^{k+r-1} - \pi_* T_{2,\delta}^{k+r-1}\right) \wedge \theta\omega^{n-k}$ is equal to

$$\int_{\mathbb{P}(E)} i\partial\overline{\partial}\theta \wedge \omega^{n-k} \wedge \left(\sum_{j=0}^{r+k-1} T_{1,\delta}^{j} \wedge T_{2,\delta}^{r+k-1-j}\right) \log\left(\frac{e^{\varphi}+\delta e^{\psi_1}}{e^{\varphi}+\delta e^{\psi_2}}\right).$$

Define

$$F_\delta := \log\left(\frac{e^{\varphi}+\delta e^{\psi_1}}{e^{\varphi}+\delta e^{\psi_2}}\right),$$

which is a uniformly bounded function on V such that $\bar{V}$ is outside of the image of the singular locus of ψ_1, ψ_2 under π. Note also that the bound is independent of δ.

Segre current

Define $Z_\eta := \{z \in V, d(z, \pi(A)) \leqslant \eta\}$ with respect to the Kähler metric ω. The volume of Z_η with respect to ω tends to 0 as $\eta \to 0$. Separate the estimate on Z_η and $X \backslash Z_\eta$. To conclude, for the first one, use Fubini theorem and that the restriction of α on each fiber is constant.

For the second one, use that F_δ tends to 0 almost everywhere as $\delta \to 0$. The convergence is locally uniform outside of the pole set A of φ.

Segre current

Corollary, -22

Let E be a strongly psef vector bundle of rank r over a compact Kähler manifold (X, ω). Let $(\mathcal{O}_{\mathbb{P}(E)}(1), h_\varepsilon)$ be singular metric with analytic singularities such that

$$i\Theta(\mathcal{O}_{\mathbb{P}(E)}(1), h_\varepsilon) \geqslant -\varepsilon\pi^*\omega$$

and the codimension of $\pi(\mathrm{Sing}(h_\varepsilon))$ is at least k in X. Then there exists a (k,k)-positive current in the class $\pi_*(c_1(\mathcal{O}_{\mathbb{P}(E)}(1)) + \varepsilon\pi^*\{\omega\})^{r+k-1}$.

Numerically flat vector bundle

Theorem, -22

Let E be a strongly psef vector bundle over a compact Kähler manifold (X, ω) with $c_1(E) = 0$. Then E is a nef vector bundle.

Idea of proof: $\exists\, T_\varepsilon \geqslant -\varepsilon\pi^*\omega \in c_1(\mathcal{O}_{\mathbb{P}(E)}(1))$ in the sense of currents with analytic singularities.
$\pi_*(T_\varepsilon + \varepsilon\pi^*\omega)^r \in c_1(E) + r\varepsilon\{\omega\} \geqslant 0$ where r is rank of E.
$c_1(E) = 0 \Rightarrow \pi_*(T_\varepsilon + \varepsilon\pi^*\omega)^r \to 0$.
Lelong number estimate $\Rightarrow$ the Lelong number of T_ε is small. We conclude by regularisation.

Application

Proposition,-22

An irreducible symplectic, or Calabi-Yau manifold does not have strongly psef tangent bundle or cotangent bundle.

In the singular and projective setting, a stronger result is proven in Theorem 1.6 of [Höring-Peternell'19] and Corollary 6.5 [Druel'18] for threefolds. (They prove that in this case $\mathcal{O}_{\mathbb{P}(E)}(1)$ is not a psef line bundle where E is the tangent bundle or the cotangent bundle.)

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$\mathbb{Q}-$conic bundle

$\mathbb{Q}-$conic bundle (Mori-Prokhorov 08)

Let X and S be normal analytic varieties. A fibration $\varphi : X \to S$ is called *a $\mathbb{Q}$-conic bundle* if it satisfies following conditions:

- X has terminal singularities;
- $\varphi : X \to S$ is equi-dimensional and of relative dimension 1;
- $-K_X$ is φ-ample.

Discriminant divisor(Mori-Prokhorov 08)

The discriminant divisor Δ is defined by the union of divisorial components of the non-smooth locus $\{s \in S \mid \varphi \text{ is not a smooth fibration at s}\}$.

classification of 3-dim $\mathbb{Q}-$conic bundle

Mori-Prokhorov 08

Let $\varphi : X \to S$ be a 3-dimensional $\mathbb{Q}$-conic bundle and $\Delta \subset S$ be the discriminant divisor. Then $s \notin \Delta$ if and only if $\varphi : X \to S$ is toroidal at s.

Example (A global $\mathbb{Q}-$conic bundle)

For a Kummer surface $S := A/\mu_2$ with a torus A of dimension 2, we consider

$$X' := (\mathbb{P}^1 \times A)/\mu_2 \to S = A/\mu_2,$$

where μ_2 acts on $\mathbb{P}^1 \times A$ by $-1 \cdot (t, z_1, z_2) = (-t, -z_1, -z_2)$. Both S and X' are simply connected and $\varphi : X' \to S$ is a $\mathbb{Q}$-conic bundle such that $-K_{X'}$ is nef. However X' is not outcome of MMP for some smooth X with $-K_X$ nef (cf. Peternell-Serrano).

Consequence

Corollary, Matsumura-Wu 23

We consider the MF space $\varphi : X' = X_N \to S$ in 3-dim Kähler MMP. Then, we have:
(1) The Bott-Chern cohomology class $-4c_1(K_S) - c_1(\Delta)$ is pseudo-effective, where Δ is the discriminant divisor of the MF space $\varphi : X \to S$ (which is a $\mathbb{Q}$-conic bundle).
(2) The relation $\Delta = 0$ and $c_1(K_S) = 0$ holds; in particular, $\varphi : X' \to S$ is toroidal over S. Furthermore, when S are smooth, the variety X is automatically smooth and $\varphi : X' \to S$ is a locally trivial $\mathbb{P}^1$-bundle.

End of proof

Let $\varphi : X' \to S$ be the MF space.

(1) Show that $\varphi_*(-pK_{X'})$ is weakly positively curved with trivial first Chern class for $1 \ll p$ by positivity of direct image.

(2) Show that $\varphi_*(-pK_{X'})$ is numerical flat orbifold vector bundle (by -23).

(3) By Campana04, S is either quotient of torus or normal K3. Show that $\varphi_*(pB)$ is trivial over some quasi-étale cover. Deduce a contradiction by intersection numbers if S is not smooth.

End of proof

In general, the positivity of direct image is insensible to singularity. It is conjectured that the fundamental group of the regular part of klt compact Kähler Calabi-Yau space is infinite if and only if it contains a torus factor in the singular Beauville-Bogomolov decomposition theorem. If this holds, Step 3 should be able to generalise to high dimensional case.

Thank you for your attention!