Compact Kähler threefolds with nef anticanonical line bundle

Xiaojun WU

Université Côte d'azur

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Ample line bundle

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Let A be a holomorphic line bundle over a projective manifold. A is called ample if for $m \gg 0$, the so-called Kodaira map

$$\Phi_{|mA|} : X \to \mathbb{P}(H^0(X, mA))$$

$$x \mapsto \{s \in H^0(X, mA); s(x) = 0\}$$

is a regular embedding.

Equivalent defintion, Kodaira '54

A is ample if and only if there exists a smooth metric on A such that its Chern curvature is strictly positive (i.e. $c_1(A)$ contains a Kähler form).

Ample line bundle

Examples of ample line bundle

(1) The anticanonical bundle $-K_{\mathbb{P}^n} := \det(T_{\mathbb{P}^n})$ of $\mathbb{P}^n$.
(2) The canonical bundle of a hypersurface in $\mathbb{P}^n$ defined by a polynomial of degree $\geqslant n+2$. (When we change the coefficients of the polynomial, we may change the complex structure of the hypersurface (called deformation).)

Kollár–Miyaoka–Mori: only finitely many deformation classes of manifolds with ample anticanonical line bundle (called Fano manifolds) of each dimension $\Longrightarrow$ Possibility of classification

General Picture of classification

Observation

$c_1(-K_X) = \{\mathrm{Ric}(\omega)\}$ where ω is any Kähler metric.

Any compact Kähler manifold is "built out" by pieces of different curvature natures with signature.
The general approach to "split" the manifolds:
Step 1: maximal rationally connected (for short, MRC) quotient $\Longrightarrow$ "split" the "Fano" part
Step 2: Core quotient (introduced by Campana '04) $\Longrightarrow$ "split" the "flat" part.
The rest is "of (log) general type". ("manifold with strictly negative curvature")
In general, the MRC/core quotient is meromorphic! $\Longrightarrow$ infinite models?

General Picture of classification

Two possible approaches:
Approach 1: Minimal Model Program
Find the "best" ones in the bimeromorphic models.
Approach 2: Given more restriction conditions, the quotients can be taken to be holomorphic or locally trivial?

Locally trivial map

A holomorphic map between complex manifolds $f : X \to Y$ is called locally trivial if for any $y \in Y$, there exists an open neighborhood U of y such that $f|_{f^{-1}(U)}$ is isomorphic to the projection map $f^{-1}(y) \times U \to U$.

Meromophic map

Definition (non-standard)

A meromorphic map $f : X \dashrightarrow Y$ between compact Kähler manifolds is a holomorphic map $f : X \backslash A \to Y$ such that A is a closed analytic subset of codimension at least 2.

A bimeromorphic map is a meromorphic map when the map defines a biholomorphism between Zariski open sets of the source and target manifolds.
MRC/core quotient is unique up to bimeromorphism.

Example: Complex surfaces

Let S be a compact Kähler surface, $R(S)$ be its MRC quotient and c_S be its core map.

Case 1: Any general point of S is contained in the image of a non-constant holomorphic map $\mathbb{P}^1 \to S$ (called a rational curve).

If $\dim(R(S)) = 0$, S is rational (i.e. bimeromorphic to $\mathbb{P}^2$).

If $\dim(R(S)) = 1$, S is a ruled surface. If $R(S)$ is an elliptic curve, c_S is constant. If the genus of $R(S)$ is at least 2, c_S is given by the MRC quotient.

Case 2: S is bimeromorphic to $R(S)$. c_S is constant iff up to some finite étale cover, S is bimeromorphic to K3 or torus or elliptic over a curve C of genus $g(C)$ with m multiple fibers with $g(C) + m \leqslant 2$.

Background

Certain "positively curved" varieties, which are often formulated to have positive holomorphic bisectional curvatures/ nefness/ pseudoeffectivity, tangent bundles, or anticanonical divisors, have occupied an important place in the classification theory.
The famous work of Siu-Yau and Mori have given a beautiful characterization of projective spaces, in terms of positive holomorphic bisectional curvatures or ample tangent bundles.
Since that time, there are huge amount of works on this subject among which Demailly-Peternell-Schneider studied the compact Kähler manifolds with nef tangent bundle and Cao-Höring studied the projective manifolds with nef anticanonical bundle.

Background

In this case, usually we hope to show that the Albanese map or a MRC quotient is a locally trivial holomorphic map over a manifold with vanishing first Chern class.
Note that MRC quotient is not unique. In general, we have to choose one in an intellgent way.

Usually, we classify the manifolds up to finite étale cover.

classification up to finite cover

Let $\Gamma = \mathbb{C}/(\mathbb{Z} + \mathbb{Z}\tau), \Im\tau > 0$ be an elliptic curve. Consider the quotient space $X = (\Gamma \times \Gamma \times \Gamma)/G$ where the action $G = \{1, g_1, g_2, g_1 g_2\} \simeq \mathbb{Z}/2 \times \mathbb{Z}/2$ is given by $g_1(z_1, z_2, z_3) = (z_1 + 1/2, -z_2, -z_3), g_2(z_1, z_2, z_3) = (-z_1, z_2 + 1/2, -z_3 + 1/2), g_1 g_2(z_1, z_2, z_3) = (-z_1 + 1/2, -z_2 + 1/2, z_3 + 1/2)$. We regard X as being built up by three pieces.

In general, a quotient of torus can be always written as the quotient of torus under an action of finite group. But the classification of such manifolds is still open in arbitrary dimension.

Nef line bundle

Nef line bundle, [DPS94]

Let (X, ω) be a compact Kähler manifold. A line bundle L over X is nef if for every $\varepsilon > 0$ there exists a smooth hermitian metric h_ε on L such that the Chern curvature satisfies

$$\Theta(h_\varepsilon) \geqslant -\varepsilon\omega.$$

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Algebraic description

If C is a curve in X and L is nef, $(L \cdot C) \geqslant 0$. It is an equivalent definition if X is projective, but not in general (e.g. $L = \mathcal{O}(-(n-1)E) = -K_X$ where E is exceptional divisor of blow-up X of a point in a very general torus of dimension n).

Nef line bundle

Conjecturally, K_X is nef iff $(K_X \cdot C) \geqslant 0$ for any curve C and K_X is pseudoeffective (i.e. $c_1(K_X)$ contains a positive current).
In general, a nef line bundle doesn't necessarily admit a smooth metric with (semi)positive curvature.

Example [DPS94]

Let C be an ellitpic curve and E be the unique non-trivial extension

$$0 \to \mathcal{O}_C \to E \to \mathcal{O}_C \to 0.$$

Then $-K_{\mathbb{P}(E)}$ is nef without any smooth metric with (semi)positive curvature.

compact Kähler manifold with nef anticanonical line bundle

Conjecture

Let X be a compact Kähler manifold with the nef anti-canonical bundle $-K_X$. Then, there exists a fibration $\varphi : X \to Y$ with the following:

- $\varphi : X \to Y$ is a locally trivial fibration;
- Y is a compact Kähler manifold with $c_1(Y) = 0$;
- F, which is the fiber of $\varphi : X \to Y$, is rationally connected (i.e. any two general points of F are in the image of some holomorphic map $\mathbb{P}^1 \to F$).

Known in projective case by Cao-Höring.
But widely open in the compact Kähler case.

Main result

Theorem, Matsumura-Wu '23

Let X be a non-projective compact Kähler 3-fold with nef anti-canonical bundle. Then X admits a finite étale cover that is one of the following:

- a compact Kähler manifold with vanishing first Chern class;
- the product of a K3 surface and the projective line $\mathbb{P}^1$;
- the projective space bundle $\mathbb{P}(E)$ of a numerical flat vector bundle E of rank 2 over a 2-dimensional (compact complex) torus.

This implies the structure conjecture in the case of $\dim X = 3$.

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Local triviality

Let $f : X \to Y$ be a map between compact Kähler manifolds.
Two approachs to show the local triviality.
(1) Intrinsic approach: Show that the fibers have no deformation of complex structure.
Starting point: Fischer-Grauert theorem '65. If f is smooth and each fiber is biholomorphic, f is locally trivial.
Difficulty: Good knowlege of moduli space in general.
Special case: If each fiber is projective line, f is locally trivial.

Local triviality

(2) Extrinsic approach: (Demailly-Peternell-Schneider, Cao-Höring)
Find a line bundle B on X such that f factorise through $X \to \mathbb{P}(f_* B) \to Y$ where the first map is a closed immersion.
Show that the fiberwise definition function of X_y in $\mathbb{P}(f_* B_y)$ has local constant coefficients when varying y.
Key concept: $f_*(B)$ is numerically flat which means that $\mathcal{O}_{\mathbb{P}(f_*(B))}(1)$ is nef and $c_1(f_*(B)) = 0$.
Difficulty: Find such a B!

Numerical flatness

Simpson's result shows that $f_*(B)$ admits a flat connection. Thus, locally, $f_*(B)$ admits a basis which is flat.
Idea to show the locally triviality:
Write the definition function of X in this basis. Assume that f is locally trivial in the smooth category. Derivative of the definition function has to be trivial. Since the basis is flat, the derivative of each coefficient is trivial by Leibniz's law.

Arguments of Cao-Höring

For simplicity, we suppose that the MRC quotient $\varphi : X \dashrightarrow R(X)$ is a holomorphic map onto a smooth projective variety $R(X)$.

(1) Construct a φ-ample line bundle B on X such that the direct image sheaf $\varphi_*(pB) := \varphi_*\mathcal{O}_X(pB)$ is weakly positively curved and satisfies that $c_1(\varphi_*(pB)) = 0$ for $1 \ll p \in \mathbb{Z}$.
Tool: Positivity of direct image

(2) Show that $\varphi_*(pB)$ is numerical flat.
Tool: Cao-Paun, Segre current

Arguments of Cao-Höring

A line bundle B is called $\varphi-$ample if there exists a Kähler form $\omega_{R(X)}$ on $R(X)$ such that $c_1(B)+\varphi^*\omega_{R(X)}$ is a Kähler class on X. Let A is an ample line bundle on X. B will be of form like $mA-\frac{1}{\mathrm{rank}(\varphi_* mA)}\mathrm{det}(\varphi_* mA)(m \gg 0)$ in Cao-Höring.
In our case, B will be $-mK_X-\frac{1}{\mathrm{rank}(\varphi_*(-mK_X))}\mathrm{det}(\varphi_*(-mK_X))$ for $m \gg 0$ since $-K_X$ is $\varphi-$ample and fibers of φ are rational curves.

Difficulty

The MRC quotient a priori is just meromorphic.
How to show $-K_X$ is relative ample?

Idea

In 3-dim compact Kähler non-projective case, the MMP consists of only one step of Mori fiber (for short, MF) space such that $-K_X$ is relative ample.

In general, MMP involves singular space...
Example (Iitaka): Let X be any resoultion of singularity of $A/\langle \pm 1 \rangle$ (A 3 dimension torus). There is no smooth manifold bimeromorphic to X with nef canonical divisor.

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Reduction

Let $R(X)$ be the MRC quotient of X.

Reduction to the case of dim $R(X) = 2$

In the case of dim $R(X) \leqslant 1$, a general fiber F of $\varphi : X \dashrightarrow R(X)$ is rationally connected, and hence has no (non-zero) holomorphic differential forms; therefore we have $h^2(X, \mathcal{O}_X) = h^0(X, \Omega_X^2) = 0$ by dim $R(X) = 1$, which implies that X is projective.
In the case of dim $R(X) = 3$, the manifold X is non-uniruled; hence K_X is pseudo-effective, which follows from [BDPP] for projective manifolds of any dimension and from [Brunella06] for compact Kähler manifolds of dimension $\leqslant 3$. This implies that $c_1(X) = c_1(K_X) = 0$ since $-K_X$ is nef.

conic bundle

Conic bundle (Sarkisov '80)

Let $\varphi : X \to S$ be a flat holomorphic map between smooth manifolds of relative dimension 1. The fibration $\varphi : X \to S$ is said to be a *conic bundle* if $-K_X$ is $\varphi-$ample. *The discriminant divisor* Δ is defined by the union of divisorial components of the non-smooth locus $\{s \in S \mid \varphi \text{ is not a smooth fibration at s}\}$.

Theorem, Matsumura-Wu 23

Let $\varphi : X \to S$ be a conic bundle with discriminant divisor Δ. Then, we have

$$\varphi_*(c_1(K_X)^2) = -4c_1(K_S) - c_1(\Delta) = \frac{4}{3}c_1(\varphi_*(-K_X)) + \frac{1}{3}c_1(\Delta).$$

Sketch of Matsumura-Wu

$-K_X$ nef $\Longrightarrow -4c_1(K_S) - c_1(\Delta) \geqslant 0$.
S is non-uniruled (Graber-Harris-Starr '03) $\Longrightarrow c_1(K_S) \geqslant 0$ (by classification of surface).
Thus $c_1(K_S) = c_1(\Delta) = 0$.
If S is smooth, Fischer-Grauert implies the local triviality.
How to show that S is smooth?

Sketch of Matsumura-Wu

By Mori-Prokhorov '08, we have a complete classification of germs of φ when dimension of X is 3 and S is singular.
If S is singular and there is no discriminant divisor, there is only one possiblity involving only quotient singularities.
The approach of Cao-Höring can be adapted in the case of quotient singularities by Matsumura-Wu '23, Wu '23.
In particular, Step (2) holds by the technique of Segre current.

Higher dimensional case?

Difficulties:
(1) $c_1(K_S)$ contains a positive current is an open conjecture in general. (The projective case is solved by Boucksom-Demailly-Paun-Peternell.)
(2) Show that the MRC quotient is a projective morphism. (Conjectured to be true by Campana and Demailly.)

Thank you for your attention!